ON THE PROBLEM OF STABILITY OF STEADY MOTIONS OF A RIGID BODY IN A POTENTIAL FORCE FIELD

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Let a rigid body with a fixed point be in a potential force field defined by a function $U = U(\gamma_1, \gamma_2, \gamma_3)$, which, for simplicity, we shall assume to be holomorphic over the set of all values that may be later required, of the variables γ_1, γ_2 and γ_3. Then, the equations of motion of the body

$$
\begin{aligned}
&A\frac{dp}{dt} = (B-C)qr + \gamma_3\frac{\partial U}{\partial \gamma_2} - \gamma_2\frac{\partial U}{\partial \gamma_3}, \qquad && \frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3 \\
&B\frac{dq}{dt} = (C-A)rp + \gamma_1\frac{\partial U}{\partial \gamma_3} - \gamma_3\frac{\partial U}{\partial \gamma_1}, && \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1 \\
&C\frac{dr}{dt} = (A-B)pq + \gamma_2\frac{\partial U}{\partial \gamma_1} - \gamma_1\frac{\partial U}{\partial \gamma_2}, && \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2
\end{aligned} \tag{1}
$$

admit the following first integrals

$$
\begin{gathered}
2H = Ap^2 + Bq^2 + Cr^2 - 2U(\gamma_1, \gamma_2, \gamma_3) = \text{const} \\
V_1 = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = \text{const}, \qquad V_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1
\end{gathered} \tag{2}
$$

Here A, B and C are moments of inertia with respect to the principal axes of the body, the origin of these axes coinciding with the fixed point; p, q and r are the projections of the angular velocity on these axes and γ_1, γ_2 and γ_3 are cosines of the angles made between the principal axes of the body and an axis which we shall call 'vertical', and which is fixed in space.

To investigate the stability of steady motions of the body we shall use, as in [3], the Routh-Liapunov theorem [4]. Let us use the integrals (2) to construct the Lagrange function

$$K = H - \lambda_1 V_1 - \tfrac{1}{2}\lambda_2 V_2 \tag{3}$$

where λ_1 and λ_2 are constant multipliers and let us write the necessary conditions for the extrema of K in terms of the variables p, q, r, γ_1, γ_2 and γ_3

$$\frac{\partial K}{\partial p} = \frac{\partial K}{\partial q} = \frac{\partial K}{\partial r} = \frac{\partial K}{\partial \gamma_i} = 0 \qquad (i = 1, 2, 3)$$

which, after obvious transformations, assume the form

$$p = \lambda_1\gamma_1, \qquad q = \lambda_1\gamma_2, \qquad r = \lambda_1\gamma_3 \tag{4}$$

$$(\lambda_1^2A + \lambda_2)\,\gamma_1 + \partial U\,/\,\partial\gamma_1 = 0,\ \ (\lambda_1^2B + \lambda_2)\,\gamma_2 + \partial U\,/\partial\gamma_2 = 0,$$
$$(\lambda_1^2C + \lambda_2)\,\gamma_3 + \partial U\,/\partial\gamma_3 = 0 \tag{5}$$

From (4) it follows, that the vertical fixed in space can be the only permanent axis and that $\lambda_1 = \omega$ will be the angular velocity of the body about this axis.

Equations (5) define, generally speaking, $\lambda_1 = \omega$ together with γ_1, γ_2 and γ_3 which define the position of the permanent axis in the body for every value of λ_2, with the relation $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ taken into account.

Eliminating λ_1 and λ_2 from (5), we obtain a surface analogous to the Staude's cone [2]

$$\frac{\partial U}{\partial\gamma_1}(B - C)\,\gamma_2\gamma_3 + \frac{\partial U}{\partial\gamma_2}(C - A)\,\gamma_3\gamma_1 + \frac{\partial U}{\partial\gamma_3}(A - B)\,\gamma_1\gamma_2 = 0 \tag{6}$$

which, together with the condition $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$, will define the set of all permanent axes in the body.

To investigate the stability of obtained motions, let us consider the condition of sign definiteness of the second variation of K on the linearised manifold defined by the integrals V_1 and V_2. We shall assume that $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2$, and η_3 are the perturbations of p, q, r, γ_1, γ_2 and γ_3, respectively. Then

$$2\delta^2K = A\xi_1^2 + B\xi_2^2 + C\xi_3^2 - 2\omega\,(A\xi_1\eta_1 + B\xi_2\eta_2 + C\xi_3\eta_3) - \lambda_2\,(\eta_1^2 + \eta_2^2 + \eta_3^2) - \tag{7}$$
$$- u_{11}\eta_1^2 - u_{22}\eta_2^2 - u_{33}\eta_3^2 - 2u_{12}\eta_1\eta_2 - 2u_{13}\eta_1\eta_3 - 2u_{23}\eta_2\eta_3$$

$$\delta V_1 = Ap\eta_1 + Bq\eta_2 + Cr\eta_3 + A\gamma_1\xi_1 + B\gamma_2\xi_2 + C\gamma_3\xi_3 + \ldots = 0$$
$$\delta V_2 = \gamma_1\eta_1 + \gamma_2\eta_2 + \gamma_3\eta_3 + \ldots = 0 \tag{8}$$

Here and in the following, we shall use the notation

$$\frac{\partial U}{\partial\gamma_i} = u_i, \qquad \frac{\partial^2 U}{\partial\gamma_i\partial\gamma_j} = u_{ij} \qquad (i,\ j = 1, 2,\ 3)$$

Following [3], let us replace ξ_i $(i = 1, 2, 3)$ with $x_i = \xi_i - \omega\eta_i$. Then, (7) and (8) become

$$2\delta^2K = Ax_1^2 + Bx_2^2 + Cx_3^2 - (A\omega^2 + \lambda_2 + u_{11})\,\eta_1^2 - (B\omega^2 + \lambda_2 + u_{22})\,\eta_2^2 - \tag{9}$$
$$- (C\omega^2 + \lambda_2 + u_{33})\,\eta_3^2 - 2u_{12}\eta_1\eta_2 - 2u_{13}\eta_1\eta_3 - 2u_{23}\eta_2\eta_3$$

$$\delta V_1 = A\gamma_1x_1 + B\gamma_2 + x_2 + C\gamma_3x_3 + 2Ap\eta_1 + 2Bq\eta_2 + 2Cr\eta_3 + \ldots = 0$$
$$\delta V_2 = \gamma_1\eta_1 + \gamma_2\eta_2 + \gamma_3\eta_3 + \ldots = 0 \tag{10}$$

The positiveness of the determinant

$$\begin{vmatrix} -A^* & -u_{12} & -u_{13} & 0 & 0 & 0 & 2Ap & \gamma_1 \\ -u_{12} & -B^* & -u_{23} & 0 & 0 & 0 & 2Bq & \gamma_2 \\ -u_{13} & -u_{23} & -C^* & 0 & 0 & 0 & 2Cr & \gamma_3 \\ 0 & 0 & 0 & A & 0 & 0 & A\gamma_1 & 0 \\ 0 & 0 & 0 & 0 & B & 0 & B\gamma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & C & C\gamma_3 & 0 \\ 2Ap & 2Bq & 2Cr & A\gamma_1 & B\gamma_2 & C\gamma_3 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

and of another three determinants obtained from it by striking out the sixth row and column, fifth and sixth row and column and finally the fourth, fifth and sixth row and column, are the conditions of sign definiteness of the quadratic form (9) on the linear manifold (10), according to well known generalisation of Silvester criterion [5 and 3]. In the above determinant, we have used for brevity

$$A^* = A\omega^2 + \lambda_2 + u_{11}, \quad B^* = B\omega^2 + \lambda_2 + u_{22}, \quad C^* = C\omega^2 + \lambda_2 + u_{33} \tag{11}$$

It can easily be checked, that the above conditions have the form

$$\begin{gathered} 4\omega^2 T_1 > 0, \qquad A\,(4\omega^2 T_1 - A\gamma_1^2 S_1) > 0 \\ AB\,(4\omega^2 T_1 - (A\gamma_1^2 + B\gamma_2^2)\,S_1) > 0 \\ ABC\,(4\omega^2 T_1 - JS_1) > 0 \qquad (J = A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) \end{gathered} \tag{12}$$

where J is the moment of inertia of the body relative to the permanent axis.

$$\begin{aligned} T_1 = &- [A^*(B-C)^2\gamma_3^2\gamma_2^2 + B^*(C-A)^2\gamma_1^2\gamma_3^2 + C^*(A-B)^2\gamma_2^2\gamma_1^2] - \\ &- 2\gamma_1\gamma_2\gamma_3\,[(A-C)(B-A)\,u_{23}\gamma_1 + (B-A)(C-B)\,u_{13}\gamma_2 + (C-B)(A-C)u_{12}\gamma_3] \end{aligned}$$

$$\begin{aligned} S_1 = &[(u_{23}^2 - B^*C^*)\,\gamma_1^2 + (u_{13}^2 - C^*A^*)\,\gamma_2^2 + (u_{12}^2 - A^*B^*)\,\gamma_3^2] - \\ &- 2\gamma_2\gamma_3\,(u_{12}u_{13} - u_{23}A^*) - 2\gamma_3\gamma_1\,(u_{23}u_{12} - u_{13}B^*) - 2\gamma_1\gamma_2\,(u_{13}u_{23} - u_{12}C^*) \end{aligned} \tag{13}$$

(14)

Obviously, whenever the first and last condition of (12) is fulfilled, the second and third must also be. Hence,

$$T_1 > 0, \qquad 4\omega^2 T_1 - JS_1 > 0 \tag{15}$$

will represent the conditions of positive definiteness of the second variation of K on the linearised manifold defined by the integrals V_1 and V_2. By the Routh-Liapunov theorem [4], they will also be sufficient conditions of stability of discussed motions in the variables p, q, r, γ_1, γ_2 and γ_3. The last condition of (15) becomes, on exclusion of the boundary $a_2 = 0$, the necessary condition, since the characteristic equation for the variational equations will, in this case, be

$$\varkappa^2\,(\varkappa^4 a_0 + \varkappa^2 a_1 + a_2) = 0 \qquad (a_0 = ABC)$$

$$\begin{aligned} a_1 = &[A\,(A-B-C)^2\gamma_1^2 + B\,(B-C-A)^2\gamma_2^2 + C\,(C-A-B)^2\gamma_3^2]\,\omega^2 - \\ &- [AA^*\,(B\gamma_2^2 + C\gamma_3^2) + BB^*\,(A\gamma_1^2 + C\gamma_3^2) + CC^*\,(B\gamma_2^2 + A\gamma_1^2)] + \\ &+ 2ABu_{12}\gamma_1\gamma_2 + 2BCu_{32}\gamma_3\gamma_2 + 2CAu_{13}\gamma_1\gamma_3 \end{aligned}$$

$$a_2 = 4\omega^2 T_1 - IS_1$$

If the force function is given in the form

$$u = x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3 + \mu\,(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$$

then conditions (15) coincide with the conditions obtained by Kuz'min [3] for the stability of permanent rotations in the central gravity field. For the force function which can be given as

$$u = f_1(\gamma_1) + f_2(\gamma_2) + f_3(\gamma_3)$$

sufficient conditions of stability can be obtained from (15) after the substitution $u_{ij} = 0$, $i \neq j$, $(i, j = 1, 2, 3)$, but they will be weaker than those obtained by Anchev in [2], for u as defined above. For the force field with the Goriachev function [6 and 7]

$$U(\gamma_1, \gamma_2, \gamma_3) = \frac{a}{n-1}\gamma_3^{1-n} + \frac{1}{2} b(\gamma_2^2 - \gamma_1^2) - c_1\gamma_1 - c_2\gamma_2$$

direction cosines γ_1, γ_2 and γ_3 of permanent axes will be, according to (5), given by

$$(A\omega^2 + \lambda_2)\gamma_1 - b\gamma_1 - c_1 = 0, \quad (B\omega^2 + \lambda_2)\gamma_2 + b\gamma_2 - c_2 = 0 \qquad (16)$$
$$(C\omega^2 + \lambda_2)\gamma_3 - a\gamma_3^{-n} = 0$$

where

$$u_1 = -b\gamma_1 - c_1, u_2 = b\gamma_2 - c_2, \quad u_3 = -a\gamma_3^{-n}, \quad u_{11} = -b, \quad u_{22} = b, \quad u_{33} = an\gamma_3^{-n-1}$$

Consequently, expressions for T_1 and S_1 will, after eliminating the parameter λ_2 by means of (16), become

$$T_1 = -\left[(B-A)^2 \frac{\gamma_1^2\gamma_2^2}{\gamma_3^{n+1}} a(n+1) + (C-A)^2 \frac{c_2\gamma_1^2\gamma_3^2}{\gamma_2} + (C-B)^2 \frac{c_1\gamma_2^2\gamma_3^2}{\gamma_1}\right]$$
$$S_1 = -\left[\frac{a(n+1)}{\gamma_3^{n+1}}\frac{c_2\gamma_1^2}{\gamma^2} + \frac{c_1\gamma_2^2}{\gamma_1} + \frac{c_1c_2\gamma_3^2}{\gamma_1\gamma_2}\right]$$

From this it follows, that sufficient conditions of stability of permanent rotations described by (16), can be written as

$$(17)$$

$$(B-A)^2 a(n+1)\gamma_1^2\gamma_2^2\gamma_3^{-(n+1)} + (C-A)^2 c_2\gamma_1^2\gamma_2^{-1}\gamma_3^2 + (C-B)^2 c_1\gamma_1^{-1}\gamma_2^2\gamma_3^2 < 0$$
$$4\omega^2 [(B-A)^2 a(n+1)\gamma_1^2\gamma_2^2\gamma_3^{-(n+1)} + (C-A)^2 c_2\gamma_1^2\gamma_2^{-1}\gamma_3^2 + (C-B)^2 c_1\gamma_1^{-1}\gamma_2^2\gamma_3^2] +$$
$$+ I\,[a(n+1)\gamma_3^{-(n+1)}(c_1\gamma_2^2\gamma_1^{-1} + c_2\gamma_1^2\gamma_2^{-1}) + c_1c_2\gamma_1^{-1}\gamma_2^{-1}\gamma_3^2] < 0$$

For the case $A = B = 2C$ investigated in [7], the above conditions simplify to

$$c_1\gamma_1^{-3} + c_2\gamma_2^{-3} < 0$$
$$A\omega^2\gamma_3^2(c_1\gamma_2^2\gamma_1^{-1} + c_2\gamma_1^2\gamma_2^{-1}) - (1 - {}^1/_2\,\gamma_3^2))\,[a(n+1)\gamma_3^{-(n+1)}(c_1\gamma_2^2\gamma_1^{-1} +$$
$$+ c_2\gamma_1^2\gamma_2^{-1}) + c_1c_2\gamma_1^{-1}\gamma_2^{-1}\gamma_3^2] < 0$$

but will be weaker than those obtained by Apykhtin in [7].

BIBLIOGRAPHY

1. Goriachev D.N., Nekotorye obshchie integraly v zadache o dvizhenii tverdogo tela (Some general integrals in the problem of motion of a rigid body). Warsaw, 1910.

2. Anchev A., O permanentnykh vrashcheniiakh tverdogo tela s odnoi nepodvizhnoi tochkoi i ikh ustoichivosti (On the stability of steady rotation of a rigid body with a fixed point). *PMM*, Vol. 29, No. 2, 1965.

3. Kuz'min P.A., Statsionarnye dvizheniia tverdogo tela i ikh ustoichivost' v tsentral'nom pole tiagoteniia. Tr. konferentsii po prikladnoi teorii ustoichivosti i analiticheskoi mekhanike (Stability of steady motions of a rigid body in a central gravity field. Trans. of a conference on applied theory of elasticity and analytical mechanics). Kazan', 1962.

4. Liapunov A.M., O postoiannykh vintovykh dvizheniiakh tverdogo tela v zhidkosti

(On steady helical motions of a rigid body in a fluid). Coll. works Vol. 1, Izd-vo Akad. Nauk SSSR, 1954.

5. Shostak R.Ia., O priznake uslovnoi opredelennosti kvadratichnoi formy n peremennykh, podchinennykh lineinym sviaziam, i o dostatochnom priznake uslovnogo ekstremuma funktsii n peremennykh (On the criterion of conditional definiteness of a quadratic form in linearly dependent n variables, and on sufficient criterion of conditional extremum of a function in n variables). Usp. matem. n., Vol. 9, No. 2, 1954.

6. Goriachev D.N., Novye sluchai integriruemosti dinamicheskikh uravnenii Eilera (New cases of integrability of Euler's dynamic equations). Khar'kov, 1916.

7. Apykhtin N.G., Ob ustoichivosti nekotorykh permanentnykh vrashchenii tverdogo tela (On the stability of some steady rotations of a rigid body). *PMM*, Vol. 29, No. 2, 1965.

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